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On the Asymptotic Behaviour of Functions of the Second Kind and Stieltjes Polynomials and on the Gauss–Kronrod Quadrature Formulas

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First we study the asymptotic behaviour on the unit circle of functions of the second kind associated with polynomials orthogonal on the unit circumference. With the help of these results we derive, as in the case of orthogonal polynomials, the asymptotic behaviour of functions of the second kind associated with polynomials orthogonal on the interval $[-1, 1]$. Special attention is given to the asymptotic behaviour on the interval $[-1, 1]$. Using the known close connection between the Stieltjes polynomials and the functions of the second kind we find that the Stieltjes polynomial $E_{n+1}(\cdot, (1-x^2)w)$ is asymptotically equal to the orthogonal polynomial $p_{n+1}(x, w)$, if $w(x)\sqrt{1-x^2}$ is positive and twice continuously differentiable on $[-1, 1]$. Furthermore we give, for sufficiently large n, several "interlacing properties" for the zeros of the Stieltjes polynomials, such as the interlacing property of the zeros of two consecutive Stieltjes polynomials, of the zeros of $E_{n+1}(\cdot, (1-x^2)w)$ and $p_n(\cdot, w)$, etc. Finally we show that for sufficiently large n the Gauss–Kronrod quadrature formula has all quadrature weights positive, if the weight function satisfies the abovementioned conditions. © 1992 Academic Press, Inc.

1. Notation and Introduction

First let us give some notation and definitions as well as some results from the theory of orthogonal polynomials on the unit circle which we need in the following. Polynomials orthogonal on the unit circle are studied in detail in the monographs of Szegö [29], Freud [3], Geronimus [10], in the survey papers of Nevai [19] and Lubinsky [14], and in [12].

Given a polynomial $P_n(z)$ of degree n, we define the *-transform by

$$P_n^*(z) := z^n \overline{P_n(1/\bar{z})}$$

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so that the coefficient of z^j in P_n^* is the complex conjugate of the coefficient of z^{n-j} in $P_n(z)$, $j=0, 1, 2, ..., n$. By

$$\phi_n(z) := \phi_n(z, d\mu) := \kappa_n z^n + \cdots$$

with

$$\kappa_n := \kappa_n(d\mu) \text{ and } \kappa_n > 0 \qquad (n=0, 1, 2, ...),$$

we denote the polynomials which are orthonormal on the unit circle $|z|=1$ with respect to the finite positive Borel measure $d\mu$ on $[-\pi, \pi]$, whose support is an infinite set,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(e^{i\theta}, d\mu)\, \overline{\phi_m(e^{i\theta}, d\mu)}\, d\mu(\theta)$$
$$= \delta_{n,m} \qquad \text{for} \quad m, n = 0, 1, 2, ...$$

Obviously we have $0 < \kappa_0 \leqslant \kappa_1 \leqslant \cdots \leqslant \kappa_n \leqslant \cdots$ which implies that

$$\lim_{n\to\infty} \kappa_n = \kappa \in (0, \infty]. \tag{1.1}$$

It is well known that these orthogonal polynomials satisfy a recurrence relation of the form

$$\kappa_n \phi_{n+1}(z, d\mu) = \kappa_{n+1} z\phi_n(z, d\mu) + \phi_{n+1}(0, d\mu)\, \phi_n^*(z, d\mu), \tag{1.2}$$

respectively

$$\kappa_n \phi_{n+1}^*(z, d\mu) = \kappa_{n+1} \phi_n^*(z, d\mu) + \overline{\phi_{n+1}(0, d\mu)}\, z\phi_n(z, d\mu), \tag{1.2'}$$

and the monic orthogonal polynomials

$$\Phi_n(z) := \Phi_n(z, d\mu) := \phi_n(z)/\kappa_n \qquad (n=0, 1, 2, ...)$$

satisfy the recurrence relation

$$\Phi_{n+1}(z, d\mu) = z\Phi_n(z, d\mu) + \Phi_{n+1}(0, d\mu)\, \Phi_n^*(z, d\mu). \tag{1.3}$$

The numbers

$$a_n := a_n(d\mu) := -\Phi_{n+1}(0, d\mu) = -\phi_{n+1}(0, d\mu)/\kappa_{n+1} \quad (n=0, 1, 2, ...) \tag{1.4}$$

are called reflection coefficients or Schur parameters. The κ_n's and a_n's are related to each other by

$$\kappa_{n+1}^2 - \kappa_n^2 = |\phi_{n+1}(0)|^2 \qquad (n=0, 1, 2, ...), \tag{1.5}$$

hence

$$\kappa_0^2/\kappa_n^2 = \prod_{\nu=0}^{n-1} (1-|a_\nu|^2) \qquad (n=0, 1, 2, ...). \tag{1.6}$$

Next let us introduce the so-called associated polynomial ψ_n of ϕ_n defined by

$$\psi_n(z, d\mu) = \frac{1}{2\pi c_0} \int_{-\pi}^{+\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} [\phi_n(e^{i\theta}) - \phi_n(z)] \, d\mu(\theta), \tag{1.7}$$

where

$$c_0 := c_0(d\mu) := \frac{1}{2\pi} \int_{-\pi}^{\pi} d\mu(\theta). \tag{1.8}$$

Note that the associated polynomials (ψ_n) satisfy the recurrence relation

$$\kappa_n(d\mu)\, \psi_{n+1}(z, d\mu) = \kappa_{n+1}(d\mu)\, z\psi_n(z, d\mu) - \phi_{n+1}(0, d\mu)\, \psi_n^*(z, d\mu) \tag{1.9}$$

and the monic polynomials (Ψ_n)

$$\Psi_{n+1}(z, d\mu) = z\Psi_n(z, d\mu) + a_n \Psi_n^*(z, d\mu), \tag{1.10}$$

i.e., the same recurrence relation as the Φ_n's with $\{a_n(d\mu)\}$ replaced by $\{-a_n(d\mu)\}$. An important relation between the ϕ_n's and ψ_n's is given by

$$\psi_n(z, d\mu)\, \phi_n^*(z, d\mu) + \phi_n(z, d\mu)\, \psi_n^*(z, d\mu) = \frac{2}{c_0} z^n \qquad (n=0, 1, 2, ...). \tag{1.11}$$

The system of associated polynomials (ψ_n) is orthonormal with respect to a measure μ^* which is defined in terms of $\mu(\theta)$ as follows (see, e.g., [27, pp. 106–107] or [22, Lemma 2]): Let

$$F(z, d\mu) := \frac{1}{2\pi c_0} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \, d\mu(\theta) \qquad \text{for} \quad |z|<1. \tag{1.12}$$

Then there is a unique measure μ^* for which

$$F(z, d\mu^*)\, F(z, d\mu) = 1 \tag{1.13}$$

and μ^* is that measure to which the associated polynomials are orthonormal. Since F and μ from (1.12) satisfy the relation (see, e.g., [26, p. 37])

$$\begin{aligned} \operatorname{Re} F(e^{i\theta}, d\mu) &:= \lim_{r\to 1^-} \operatorname{Re}\{F(re^{i\theta}, d\mu)\} \\ &= \mu'(\theta)/c_0 \qquad \text{for} \quad \theta \in [a, b], \end{aligned} \tag{1.14}$$

if μ is absolutely continuous on $[a, b] \subset [-\pi, \pi]$ and μ' is continuous on $[a, b]$ it follows that, under the additional condition of the positiveness of μ' on $[a, b]$, μ and μ^* are related to each other on $[a, b]$ by

$$\begin{aligned} \mu^{*\prime}(\theta)/c_0^* &= \operatorname{Re}\{1/F(e^{i\theta}, d\mu)\} \\ &= \mu'(\theta)/c_0 \, |F(e^{i\theta}, d\mu)|^2 \qquad \text{for} \quad \theta \in [a, b]. \end{aligned} \tag{1.15}$$

Furthermore, let us mention that (see [7, Theorem 13.1])

$$\lim_{n \to \infty} \frac{\psi_n^*(z, d\mu)}{\phi_n^*(z, d\mu)} = F(z, d\mu) \qquad \text{uniformly for} \quad |z| \leqslant r < 1. \tag{1.16}$$

Next let us define the nth function of the second kind $g_n(z, d\mu)$ on the unit circle by, for $n \in \mathbb{N}$ (henceforth $\mathbb{N} := \{1, 2, 3, ...\}$ and $\mathbb{N}_0 := \{0, 1, 2, ...\}$),

$$\begin{aligned} g_n(z, d\mu) &:= \frac{1}{2\pi c_0} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \phi_n(e^{i\theta}, d\mu) \, d\mu(\theta) \\ &= \phi_n(z, d\mu) \, F(z, d\mu) + \psi_n(z, d\mu) \\ &= (2/c_0 \kappa_n) \, z^n + O(z^{n+1}), \qquad |z| < 1, \end{aligned} \tag{1.17}$$

and the function $h_n(z, d\mu)$, which we call the nth associated function of second kind, by

$$\begin{aligned} h_n(z, d\mu) &:= \frac{z^n}{2\pi c_0} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \overline{\phi_n(e^{i\theta}, d\mu)} \, d\mu(\theta) \\ &= \phi_n^*(z, d\mu) \, F(z, d\mu) - \psi_n^*(z, d\mu) \\ &= (2a_n/c_0 \kappa_n) \, z^{n+1} + O(z^{n+2}), \qquad |z| < 1 \end{aligned} \tag{1.18}$$

(the second and third equality in relation (1.17) and (1.18) follow from [7, pp. 16, 35]). In the study of the asymptotic behaviour of the nth function of the second kind it will turn out that it is of advantage to consider the functions

$$\tilde{g}_n(z, d\mu) = z^{-n} g_n(z, d\mu) \qquad \text{and} \qquad \tilde{h}_n(z, d\mu) = z^{-(n+1)} h_n(z, d\mu) \tag{1.19}$$

instead of g_n and h_n. Furthermore, the so-called Szegö function plays an important role. For a nonnegative measurable function v in $[-\pi, \pi]$ satisfying the Szegö condition $\log v \in L_1[-\pi, \pi]$ the Szegö function $D(z, v)$ is defined by

$$D(z, v) := \exp\left\{\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log v(\theta) \, d\theta\right\}, \qquad |z| < 1, \tag{1.20}$$

and has the following properties (see, e.g., [10, pp. 13–19, 29, Chap. X]): $D(z, v) \neq 0$ for $|z| < 1$, $D(0, v) > 0$, $D(v) \in H_2$, whence the radial boundary values

$$D(e^{i\theta}, v) := \lim_{r \to 1^-} D(re^{i\theta}, v) = \sqrt{v(\theta)}\, e^{-i\gamma(\theta, v)}, \tag{1.21}$$

where $\gamma(\theta, v) = 2^{-1} \log \tilde{v}(\theta)$ exist almost everywhere in $[-\pi, \pi]$, $\tilde{v}(\theta)$ denotes the conjugate function to $v(\theta)$, i.e.,

$$\tilde{v}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(\tau) \operatorname{ctg} \frac{\theta - \tau}{2}\, d\tau,$$

and the integral is defined as a Cauchy principal value. Note that by (1.21)

$$|D(e^{i\theta}, v)|^2 = v(\theta) \qquad \text{a.e. in } [-\pi, \pi] \tag{1.22}$$

and in the case when v is continuous and positive on $[a, b] \subset [-\pi, \pi]$

$$|D(e^{i\theta}, v)|^2 = v(\theta) \qquad \text{for all} \quad \theta \in [a, b]. \tag{1.22'}$$

Next let us state some facts on the connection between polynomials orthonormal on $[-1, 1]$ and polynomials orthonormal on the unit circle. Let $d\alpha$ be a finite positive Borel measure on $[-1, 1]$ whose support is an infinite set and let

$$p_n(x, d\alpha) = k_n x^n + \cdots \text{with } k_n := k_n(d\alpha), \qquad n = 0, 1, 2, ...,$$

be the polynomials orthonormal with respect to $d\alpha$ on $[-1, 1]$, i.e.,

$$\int_{-1}^{1} p_n(x, d\alpha)\, p_m(x, d\alpha)\, d\alpha(x) = \delta_{n,m} \qquad \text{for} \quad n, m \in \mathbb{N}_0.$$

$P_n(x, d\alpha) = x^n + \cdots$, $n \in \mathbb{N}_0$, denotes the monic orthogonal polynomial. To such a measure α on $[-1, 1]$ we attach the measure

$$\mu(\theta) = \begin{cases} \alpha(1) - \alpha(\cos\theta) & \text{for} \quad \theta \in [0, \pi], \\ \alpha(\cos\theta) - \alpha(1) & \text{for} \quad \theta \in [-\pi, 0] \end{cases} \tag{1.23}$$

on the unit circle. Obviously, if α is absolutely continuous on $[-1, 1]$ with $\alpha'(x) = w(x)$ then μ is absolutely continuous on $[-\pi, \pi]$ with

$$\mu'(\theta) = w(\cos\theta)\, |\sin\theta| \qquad \text{for} \quad \theta \in [-\pi, \pi]. \tag{1.24}$$

If we set $z = y - \sqrt{y^2 - 1}$ for $y \in \mathbb{C} \backslash [-1, 1]$ we have the relationship between the Stieltjes transform of α and the function $F(\cdot, d\mu)$ from (1.12) (see, e.g., [7, p. 64])

$$\frac{\sqrt{y^2 - 1}}{s_0} \int_{-1}^{1} \frac{d\alpha(x)}{y - x} = F(z, d\mu), \tag{1.25}$$

where

$$s_0 := s_0(d\alpha) = \int_{-1}^{1} d\alpha(x) = \pi c_0(d\mu). \tag{1.26}$$

Furthermore we attach to the measure α the measure

$$\alpha^*(\cos\theta) = -\mu^*(\theta) \qquad \text{for} \quad \theta \in [0, \pi], \tag{1.27}$$

where μ^* is related to the measure μ from (1.23) by relation (1.13). Putting

$$A_n := A_n(d\mu) := \sqrt{2\pi(1 + a_{2n-1}(d\mu))}$$

and (1.28)

$$B_n := B_n(d\mu) := \sqrt{2\pi(1 - a_{2n-1}(d\mu))}$$

the following relations hold for $n \in \mathbb{N}$, $x = \frac{1}{2}(z + z^{-1})$ (see [29, p. 294] and [7, p. 65]):

$$p_n(x, d\alpha) = A_n^{-1}\{z^{-n+1}\phi_{2n-1}(z, d\mu) + z^{n-1}\phi_{2n-1}(z^{-1}, d\mu)\}, \tag{1.29}$$

$$p_n(x, d\alpha^*) = B_n^{-1}\{z^{-n+1}\psi_{2n-1}(z, d\mu) + z^{n-1}\psi_{2n-1}(z^{-1}, d\mu)\}, \tag{1.30}$$

$$p_{n-1}(x, (1-x^2)\, d\alpha) = 2B_n^{-1} \frac{\{z^{-n+1}\phi_{2n-1}(z, d\mu) - z^{n-1}\phi_{2n-1}(z^{-1}, d\mu)\}}{z - z^{-1}}, \tag{1.31}$$

and

$$p_{n-1}^{(1)}(x, d\alpha) = 2A_n^{-1} \frac{\{z^{-n+1}\psi_{2n-1}(z, d\mu) - z^{n-1}\psi_{2n-1}(z^{-1}, d\mu)\}}{z - z^{-1}}, \tag{1.32}$$

where $p_{n-1}^{(1)}(x, d\alpha)$ denotes the associated polynomial of $p_n(x, d\alpha)$, i.e.,

$$p_{n-1}^{(1)}(x, d\alpha) = \frac{1}{s_0} \int_{-1}^{1} \frac{p_n(x, d\alpha) - p_n(t, d\alpha)}{x - t}\, d\alpha(t). \tag{1.33}$$

Note that by (1.31) and (1.32)

$$p_{n-1}^{(1)}(x, d\alpha) = p_{n-1}(x, (1-x^2)\, d\alpha^*). \tag{1.34}$$

Finally the function of the second kind with respect to the measure α is defined by

$$q_n(y, d\alpha) = \frac{1}{s_0} \int_{-1}^{1} \frac{p_n(x, d\alpha)}{y - x}\, d\alpha(x) \qquad \text{for} \quad y \in \mathbb{C} \backslash [-1, 1] \tag{1.35}$$

and, as usual, we put for $y \in [-1, 1]$

$$q_n(y, d\alpha) = \lim_{\varepsilon \to 0^+} \tfrac{1}{2} [q_n(y + i\varepsilon, d\alpha) + q_n(y - i\varepsilon, d\alpha)] \tag{1.36}$$

and hence, for $y \in (-1, 1)$,

$$q_n(y, d\alpha) = \frac{1}{s_0} ⨍_{-1}^{1} \frac{p_n(x, d\alpha)}{y - x} \, d\alpha(x), \tag{1.37}$$

where $⨍$ denotes the Cauchy principal value.

In 1894 Stieltjes introduced and studied for the Legendre weight $w(x) := \alpha'(x) = 1$ (his methods work also for general distributions $d\alpha$) those polynomials $E_{n+1}(y, d\alpha) = y^{n+1} + \cdots$, nowaday called Stieltjes polynomials, which are the polynomial part of the series expansion of $\{s_0 k_n q_n(y, d\alpha)\}^{-1}$ at the point infinity, i.e.,

$$\{s_0 k_n q_n(y, d\alpha)\}^{-1} = E_{n+1}(y, d\alpha) + \frac{d_1}{y} + \frac{d_2}{y^2} + \cdots. \tag{1.38}$$

He showed (for a simple proof see, e.g., [24, Lemma 1]) that relation (1.38) is equivalent to the orthogonality condition

$$\int_{-1}^{1} x^j E_{n+1}(x, d\alpha) \, p_n(x, d\alpha) \, d\alpha(x) = 0 \qquad \text{for} \quad j = 0, ..., n. \tag{1.39}$$

Naturally the following questions arise: (i) for which distributions $d\alpha$ are the zeros of $E_{n+1}(\cdot, d\alpha)$ real, simple, and contained in the interval $(-1, 1)$; and (ii) do the zeros of $E_{n+1}(\cdot, d\alpha)$ and $p_n(\cdot, d\alpha)$ separate each other? For the Legendre weight Stieltjes conjectured that the polynomials E_{n+1} have these two properties. In 1934, G. Szegö [28] proved Stieltjes' conjecture. In addition he proved that the conjecture holds true for the Gegenbauer weight function $w(x, \lambda) = (1 - x^2)^{\lambda - 1/2}$, $0 < \lambda \leqslant 2$.

Let us also note that Geronimus [6] in 1929, apparently unaware of Stieltjes' results, considered polynomials $J_n(y, d\alpha) = y^{n+1} + \cdots$ defined by

$$1/s_0 k_n \sqrt{y^2 - 1} \, q_n(y, d\alpha) = J_n(y, d\alpha) + \frac{e_1}{y} + \frac{e_2}{y^2} + \cdots. \tag{1.40}$$

It turned out, see [18, Theorem 2] or Lemma 4.1 in this paper, that there is a close connection between Stieltjes and Geronimus polynomials.

The interest in Stieltjes polynomials was renewed when Kronrod introduced in 1964 the quadrature formula, now called the Gauss–Kronrod quadrature formula,

$$\int_{-1}^{1} f(x) \, d\alpha(x) = \sum_{\nu=1}^{n} \sigma_{\nu,n} f(x_{\nu,n}) + \sum_{\mu=1}^{n+1} \gamma_{\mu,n} f(y_{\mu,n}) + R_n(f), \tag{1.41}$$

where $x_{\nu,n}$ are the zeros of $p_n(\cdot, d\alpha)$ and the nodes $y_{\mu,n}$ and weights $\sigma_{\nu,n}, \gamma_{\mu,n}$ are chosen to maximize the degree of exactness of (1.41); thus $R_n(f)=0$ for all $f \in \mathbb{P}_{3n+1}$ ($\mathbb{P}_n$ denotes the set of polynomials of degree at most n) at least. It is not hard to demonstrate that the exactness condition $R_n(f)=0$ for $f \in \mathbb{P}_{3n+1}$ is equivalent to the fact that $\prod_{\mu=1}^{n+1}(x-y_{\mu,n})$ satisfies the orthogonality condition (1.39) and thus, by the uniqueness of the polynomial satisfying (1.39) which follows by the equivalence of (1.38) and (1.39),

$$E_{n+1}(x, d\alpha) = \prod_{\mu=1}^{n+1}(x-y_{\mu,n}).$$

Surveys on Stieltjes polynomials and Gauss–Kronrod quadrature formulas have been given by Monegato [18] and Gautschi [4]. Concerning the Kronrod quadrature formula (1.41) the following question, besides (i) and (ii), is of interest: (iii) for which distributions $d\alpha$ are all quadrature weights $\sigma_{\nu,n}$ and $\gamma_{\mu,n}$ positive? If the answer to question (ii) is positive, question (iii) requires in fact the positiveness of the $\sigma_{\nu,n}$'s only, since by Monegato [16] the positiveness of the $\gamma_{\mu,n}$'s is equivalent to the interlacing property of the zeros of $E_{n+1}(x, d\alpha)$ and $p_n(x, d\alpha)$. With the help of Szegö's result [28], Monegato [17] has shown that the Gegenbauer weight $w(x, \lambda)=(1-x^2)^{\lambda-1/2}$ has all quadrature weights in (1.41) positive for $0<\lambda\leqslant 1$. Notaris [21] and, independently, the author [23] have shown that weight functions of the form

$$W(x, s_m) := \sqrt{1-x^2}/s_m(x) \qquad \text{for} \quad x\in(-1, 1),$$

where s_m is a polynomial of degree m positive on $[-1, 1]$, also satisfy properties (i)–(iii) for $n \geqslant m$. For the special case $s_2(x)=(1+\lambda)^2-4\lambda x^2$, $-1<\lambda\leqslant 1$, this was first discovered by Gautschi and Rivlin [5]. Recently we succeeded in proving that weight functions having a representation of the form

$$W(x)=\sqrt{1-x^2}\,|f(e^{i\theta})|^2, \qquad x=\cos\theta, \theta\in[0,\pi],$$

where $f(z)$ is analytic and $f(z)\neq 0$ for $|z|\leqslant 1$, satisfy all three properties too for sufficiently large n. In this paper, using a different approach, we extend this result to the wide class of weight functions of the form

$$W(x)=(1-x^2)\,w(x) \tag{1.42}$$

satisfying

$$\sqrt{1-x^2}\,w(x)\in C^2[-1, 1]$$

and (1.42′)

$$\sqrt{1-x^2}\,w(x)>0 \qquad \text{for} \quad x\in[-1, 1].$$

The crucial point is the derivation of an asymptotic formula (respectively of a local asymptotic formula if w satisfies (1.42′) on a subinterval $[\zeta_1, \zeta_2] \subset [-1, 1]$ only) for the Stieltjes polynomial E_{n+1}. In order to get this asymptotic formula for E_{n+1} one needs the asymptotic behaviour of the nth function of the second kind on the cut $[-1, +1]$. Similarily, as in the study of asymptotics of orthogonal polynomials on $[-1, 1]$ (see, e.g., [29, Chap. IX]), it turns out that the asymptotic behaviour of the function of the second kind on $[-1, 1]$ can be derived from the asymptotic behaviour of the function of the second kind and its associated function on the unit circle.

This paper is organized as follows: In Section 2 the asymptotic behaviour of the function of second kind on the unit circle and its associated function is studied. In Section 3 it is shown how functions of the second kind on the unit circle and on the interval $[-1, 1]$ are related to each other. With the help of these results asymptotics for the function of the second kind on $[-1, 1]$ are derived. In Section 4 for weight functions of the form (1.42), satisfying the conditions (1.42′), an asymptotic formula for the Stieltjes polynomial is presented; interlacing properties of zeros, as those of $E_{n+1}(\cdot, W)$ and $p_n(\cdot, w)$, $E_{n+1}(\cdot, W)$ and $E_n(\cdot, W)$, are given; and finally the positiveness of the Kronrod-quadrature weights is shown.

2. On the Asymptotic Behaviour of Functions of the Second Kind on the Unit Circle

Since, as we shall see, the asymptotic behaviour of functions of the second kind is closely related to that of the polynomials ϕ_n^* and ψ_n^*, let us first state some known facts on the asymptotic behaviour of orthonormal polynomials on the unit circle. One of the important results is the equivalence of the following four statements (see, e.g., [7, Theorem 21.1; 10, p. 9]): (i) The absolutely continuous part μ' of μ satisfies Szegö's condition, i.e., $\log \mu' \in L^1[-\pi, \pi]$; (ii) The finite limit $\lim_{n\to\infty} \kappa_n(d\mu) = \kappa$ exists; (iii) $\sum_{n=0}^{\infty} |a_n(d\mu)|^2 < \infty$; (iv) $\{\phi_n^*(z, d\mu)\}$ converges uniformly for $|z| \leqslant r < 1$. In the case of convergence of $\{\phi_n^*(z, d\mu)\}$ we have

$$\lim_{n\to\infty} \phi_n^*(z, d\mu) = D(z, \mu')^{-1} \qquad \text{uniformly on} \quad |z| \leqslant r < 1, \tag{2.1}$$

where D is defined in (1.20), and in particular,

$$\lim_{n\to\infty} \kappa_n(d\mu) = D(0, \mu')^{-1}. \tag{2.2}$$

Furthermore let us note that (2.1) immediately implies

$$\lim_{n\to\infty} z^{-n}\phi_n(z, d\mu) = 1/\overline{D\left(\frac{1}{\bar{z}}, \mu'\right)} \qquad \text{uniformly on} \quad |z| \geqslant R > 1 \tag{2.3}$$

and, with the help of [10, (1.12)],

$$\lim_{n\to\infty} \phi_n(z, d\mu) = 0 \qquad \text{uniformly on} \quad |z| \leqslant r < 1. \tag{2.4}$$

It is also worth mentioning that (2.4), and moreover (2.1), together with (1.1) and (1.4), implies

$$\lim_{n\to\infty} a_n(d\mu) = 0. \tag{2.5}$$

Concerning the uniform convergence on the whole circumference $|z| = 1$ we have the following result which is essentially due to Szegö (see [1, 12]): if

$$\mu' \in C_{2\pi},\ \mu'(\tau) > 0 \text{ for } \tau \in \mathbb{R},\ \omega(\tau, \mu')/\tau \in L^1[0, \pi], \tag{2.6}$$

then

$$\phi_n^*(e^{i\theta}, d\mu) = D(e^{i\theta}, \mu')^{-1} + \varepsilon_n(e^{i\theta}, d\mu), \tag{2.7}$$

where

$$\lim_{n\to\infty} \varepsilon_n(e^{i\theta}, d\mu) = 0 \qquad \text{uniformly for} \quad \theta \in [-\pi, \pi];$$

$\omega(\tau, f)$ denotes the modulus of continuity of a function $f \in C_{2\pi}$, i.e.,

$$\omega(\tau, f) = \max\{|f(\delta) - f(\theta)| : \delta, \theta \in \mathbb{R}, |\delta - \theta| \leqslant \tau\}.$$

If $\omega(\tau, \mu') \leqslant K_1 \tau^\alpha$, $0 < \alpha \leqslant 1$, or $\omega(\tau, \mu') \leqslant K_2 |\log \tau|^{-1-\lambda}$, $\lambda > 0$, then (see [12, p. 53]) the error function ε_n from (2.7) satisfies

$$|\varepsilon_n(e^{i\theta}, d\mu)| \leqslant K_3 \omega\left(\frac{\pi}{n}, \mu'\right) \log n \qquad \text{for} \quad \theta \in [-\pi, \pi]. \tag{2.8}$$

Now, set for $f \in C[a, b]$, $[a, b] \subset [0, 2\pi]$,

$$\omega(\tau, f)_{[a,b]} = \max\{|f(\delta) - f(\theta)| : \delta, \theta \in [a, b], |\theta - \delta| \leqslant \tau\}.$$

For an arc (e^{ia}, e^{ib}) of the unit circumference Badkov [1] has shown that the limit relation (2.7) holds for all $\theta \in (a, b) \subset [-\pi, \pi]$ and uniformly inside (a, b), i.e., uniformly on each subinterval $[a_1, b_1] \subset (a, b)$, if μ satisfies the conditions

$$\log \mu' \in L^1[-\pi, \pi],\ \mu' \in C[a, b],\ \mu' > 0 \text{ on } [a, b]$$

and (2.9)

$$\omega(\tau, \mu')_{[a,b]}/\tau \in L^1[0, b-a].$$

Under more restrictive conditions on μ an estimation for the rate of convergence can be found in [10, Theorem 5.6]. For the following we need

LEMMA 2.1. *The following relations hold*:

$$\phi_n^*(z)\,\tilde{g}_n(z)-z\phi_n(z)\,\tilde{h}_n(z)=\frac{2}{c_0} \tag{2.10}$$

$$\kappa_n z\tilde{h}_{n+1}(z)=\kappa_{n+1}\tilde{h}_n(z)+\overline{\phi_{n+1}(0)}\,\tilde{g}_n(z), \tag{2.11}$$

$$\kappa_n\,\tilde{g}_{n+1}(z)=\kappa_{n+1}\,\tilde{g}_n(z)+\phi_{n+1}(0)\,\tilde{h}_n(z), \tag{2.12}$$

$$\tilde{h}_{n+1}(z)\,\overline{\tilde{h}_{n+1}(\xi)}-\tilde{g}_{n+1}(z)\,\overline{\tilde{g}_{n+1}(\xi)}-(\tilde{h}_0(z)\,\overline{\tilde{h}_0(\xi)}-\tilde{g}_0(z)\,\overline{\tilde{g}_0(\xi)})$$
$$=(1-z\bar{\xi})\sum_{j=1}^{n+1}\tilde{h}_j(z)\,\overline{\tilde{h}_j(\xi)}\qquad \text{for}\quad z,\xi\in\mathbb{C}. \tag{2.13}$$

Proof. Using the representations of g_n and h_n given in the second equality in (1.17), resp. (1.18), relation (2.10) follows immediately from (1.11). Relations (2.11) and (2.12) follow from the recurrence relations (1.2) and (1.9) of ϕ_{n+1} and ψ_{n+1}.

Thus only relation (2.13) remains to be proved. In view of (2.11) and (2.12) we have

$$\kappa_n z\tilde{h}_{n+1}(z)=\kappa_{n+1}\tilde{h}_n(z)+\overline{\phi_{n+1}(0)}\,\tilde{g}_n(z),$$
$$\kappa_n\,\tilde{g}_{n+1}(z)=\kappa_{n+1}\,\tilde{g}_n(z)+\phi_{n+1}(0)\,\tilde{h}_n(z),$$
$$\kappa_n\overline{\xi\tilde{h}_{n+1}(\xi)}=\kappa_{n+1}\overline{\tilde{h}_n(\xi)}+\phi_{n+1}(0)\,\overline{\tilde{g}_n(\xi)},$$
$$\kappa_n\,\overline{\tilde{g}_{n+1}(\xi)}=\kappa_{n+1}\,\overline{\tilde{g}_n(\xi)}+\overline{\phi_{n+1}(0)}\,\overline{\tilde{h}_n(\xi)}.$$

Multiplying and subtracting we get with the help of (1.5) that

$$\tilde{h}_{n+1}(z)\,\overline{\tilde{h}_{n+1}(\xi)}-\tilde{g}_{n+1}(z)\,\overline{\tilde{g}_{n+1}(\xi)}$$
$$=\tilde{h}_n(z)\,\overline{\tilde{h}_n(\xi)}-\tilde{g}_n(z)\,\overline{\tilde{g}_n(\xi)}+(1-z\bar{\xi})\,\tilde{h}_{n+1}(z)\,\overline{\tilde{h}_{n+1}(\xi)}$$

from which by induction relation (2.13) follows. ∎

Relation (2.13) can be considered as a Christoffel–Darboux formula for functions of the second kind.

LEMMA 2.2. *The nth function of the second kind $\tilde{g}_n$ and its associated function $\tilde{h}_n$, $n\in\mathbb{N}_0$, are analytic on $|z|<1$. Furthermore for each $r\in(0,1)$ the sequences $\{\tilde{g}_n(z)\}$ and $\{\tilde{h}_n(z)\}$ are uniformly bounded on $|z|\leqslant r$.*

Proof. Using the orthogonality property of ϕ_n we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(z^{-1}+e^{-i\theta})\left(\frac{z^{-m}-e^{-im\theta}}{z^{-1}-e^{-i\theta}}\right)\phi_n(e^{i\theta})\,d\mu = \begin{cases}0 & \text{for } 1\leqslant m\leqslant n-1,\\ 1/\kappa_n^2 & \text{for } m=n,\end{cases} \tag{2.14}$$

which implies, by (1.17), that for $|z|<1$

$$\tilde{g}_n(z)=\frac{1}{2\pi c_0}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}e^{-in\theta}\phi_n(e^{i\theta})\,d\mu+1/\kappa_n^2 c_0 \tag{2.15}$$

and hence $\tilde{g}_n(z)$ is analytic on $|z|<1$. Since by (2.15) for $|z|\leqslant r<1$

$$|\tilde{g}_n(z)|\leqslant\frac{(1+r)}{(1-r)}\frac{1}{\sqrt{c_0}}\sqrt{\frac{1}{2\pi}\int_{-\pi}^{+\pi}|\phi_n(e^{i\theta})|^2\,d\mu}+1/\kappa_n^2 c_0$$

the uniform boundedness of $\{\tilde{g}_n(z)\}$ on $|z|\leqslant r<1$ follows with the help of (1.1).

The analyticity of $\tilde{h}_n$ follows immediately from the fact that again by the orthogonality property of ϕ_n

$$\frac{z^{-1}}{2\pi c_0}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\overline{\phi_n(e^{i\theta})}\,d\mu=\frac{1}{\pi c_0}\int_{-\pi}^{\pi}\frac{\overline{\phi_n(e^{i\theta})}}{e^{i\theta}-z}\,d\mu.$$

The uniform boundedness of $\{\tilde{h}_n(z)\}$ on $|z|\leqslant r<1$ can be derived now quite similarily to that of $\{\tilde{g}_n(z)\}$. ▌

The following theorem gives a description of the asymptotic behaviour of the functions of the second kind on the unit circle.

THEOREM 2.1. (a) *Both sequences* $\{\tilde{h}_n(z, d\mu)\}$ *and* $\{\tilde{g}_n(z, d\mu)/\kappa_n\}$ *converge uniformly on* $|z|\leqslant r<1$, *where*

$$\lim_{n\to\infty}\tilde{h}_n(z, d\mu)=0 \qquad \textit{for} \quad |z|<1. \tag{2.16}$$

If the finite limit $\lim_{n\to\infty}\kappa_n(d\mu)=\kappa$ *exists, then*

$$\lim_{n\to\infty}\tilde{g}_n(z, d\mu)=\frac{2}{c_0}D(z, \mu') \qquad \textit{uniformly on} \quad |z|\leqslant r<1, \tag{2.17}$$

where μ' *is the absolutely continuous part of* μ.

(b) *Let* $R\geqslant 1$. *Suppose that both limit relations*

$$\lim_{n\to\infty}\phi_n^*(z, d\mu)=D(z, \mu')^{-1} \qquad \textit{and} \qquad \lim_{n\to\infty}\psi_n^*(z, d\mu)=D(z, (\mu^*)')^{-1} \tag{2.18}$$

hold uniformly for $|z| = R$, *respectively, uniformly on* $|z| \leqslant r < R$ *and for all* $z = Re^{i\theta}$ *on the arc* $[Re^{ia}, Re^{ib}]$, *where convergence is uniform on each subarc* $[Re^{ia_1}, Re^{ib_1}] \subset (Re^{ia}, Re^{ib})$. *Then the functions* $\tilde{g}_n$ *and* $\tilde{h}_n$ *have an analytic continuation in* $|z| < R$ *and are continuous in* $|z| \leqslant R$, *respectively in* $|z| < R$ *and in* (Re^{ia}, Re^{ib}), *and both limit relations*

$$\lim_{n\to\infty} \tilde{g}_n(z, d\mu) = \frac{2}{c_0} D(z, \mu') \qquad \text{and} \qquad \lim_{n\to\infty} \tilde{h}_n(z, d\mu) = 0 \tag{2.19}$$

as well as uniform convergence hold on the same set as in (2.18).

Proof. (a) In view of Lemma 2.1 (relation (2.13)) we have

$$\frac{|\tilde{h}_{n+1}(z)|^2 - |\tilde{g}_{n+1}(z)|^2 - (|\tilde{h}_0(z)|^2 - |\tilde{g}_0(z)|^2)}{1-|z|^2} = \sum_{j=1}^{n+1} |\tilde{h}_j(z)|^2. \tag{2.20}$$

Since by Lemma 2.2 $\{\tilde{h}_n\}$ and $\{\tilde{g}_n\}$ are uniformly bounded on $|z| \leqslant r < 1$ it follows that the series on the right hand side in (2.20) converges, which implies that

$$\lim_{n\to\infty} \tilde{h}_n(z) = 0 \qquad \text{uniformly on} \quad |z| \leqslant r < 1.$$

Using the fact that again by (2.13)

$$\overline{\tilde{g}_0(0)}\, \tilde{g}_0(z) - \overline{\tilde{g}_{n+1}(0)}\, \tilde{g}_{n+1}(z) = \sum_{j=0}^{n} \overline{\tilde{h}_j(0)}\, \tilde{h}_j(z) \tag{2.21}$$

and hence

$$\begin{aligned} &|\tilde{g}_{n+1}(0)\, \tilde{g}_{n+1}(z) - \tilde{g}_m(0)\, \tilde{g}_m(z)| \\ &= \left| \sum_{j=m}^{n} \overline{\tilde{h}_j(0)}\, \tilde{h}_j(z) \right| \\ &\leqslant \sqrt{\sum_{j=m}^{n} |\tilde{h}_j(0)|^2} \sqrt{\sum_{j=m}^{n} |\tilde{h}_j(z)|^2} \end{aligned}$$

we obtain, noting that by (1.17) $\overline{\tilde{g}_n(0)} = 2\kappa_0^2/\kappa_n$, the uniform convergence of $\{\tilde{g}_n(z)/\kappa_n\}$ on $|z| \leqslant r < 1$.

Concerning limit relation (2.17) let us recall (see the beginning of this section) that $\lim_{n\to\infty} \kappa_n = \kappa \in \mathbb{R}$ implies that limit relation (2.4) holds. Combining relations (2.10), (2.1), (2.16), and (2.4) the limit relation (2.17) follows.

(b) First let us consider the case where the limit relations in (2.18) hold uniformly for $|z| = R$ and thus uniformly for $|z| \leqslant R$. Since

$$\lim_{n\to\infty} \frac{\psi_n^*(z, d\mu)}{\phi_n^*(z, d\mu)} = \frac{D(z, (\mu^*)')}{D(z, \mu')} \tag{2.22}$$

it follows immediately by Weierstraß's Theorem that $D(z, (\mu^*)')/D(z, \mu')$ is analytic in $|z| < R$ and, of course, continuous on $|z| = R$. Hence, recalling (1.16), $D(z, (\mu^*)')/D(z, \mu')$ is the analytic, respectively continuous continuation of F in $|z| < R$, respectively $|z| = R$. In view of the second equality in (1.17) and (1.18) and Lemma 2.2 the statement on the analytic, respectively continuous continuation of $\tilde{h}_n$ and $\tilde{g}_n$ is proved.

Next let observe that by (1.18) and (2.22)

$$\lim_{n \to \infty} \tilde{h}_n(z) = 0 \qquad \text{uniformly on} \quad 1 \leqslant |z| \leqslant R \tag{2.23}$$

and thus

$$\lim_{n \to \infty} \tilde{h}_n(z) = 0 \qquad \text{uniformly on} \quad |z| \leqslant R,$$

which proves one part of (b). Since by the first limit relation in (2.18) $\{\overline{\phi_n^*(e^{i\theta})} = e^{-in\theta}\phi_n(e^{i\theta})\}$ converges uniformly for $\theta \in [-\pi, \pi]$ and thus, by (2.3), $\{z^{-n}\phi_n(z)\}$ converges uniformly for $|z| \geqslant 1$, we obtain from relation (2.10), taking into account (2.23), the convergence behaviour of $\{\tilde{g}_n(z)\}$.

The remaining case, when convergence on a subarc is given, can be demonstrated quite similarily. ▮

Let us note that by (2.17) and (2.21)

$$\sum_{j=0}^{\infty} \overline{h_j(0)}\, \tilde{h}_j(z) = 2\kappa_0^2 \left[1 + F(z, d\mu) - \frac{2\kappa_0^2}{\kappa} D(z, \mu')^{-1} \right] \tag{2.21'}$$

uniformly on $|z| \leqslant r < 1$, if the finite limit $\lim_{n \to \infty} \kappa_n(d\mu) = \kappa$ exists.

For the special case (see [7, p. 24]) that the reflection coefficients $\{a_\nu\}$ satisfy

$$|a_\nu| < 1 \text{ for } \nu = 0, 1, ..., m-1 \qquad \text{and} \qquad a_\nu = 0 \qquad \text{for} \quad \nu = m, m+1, ...,$$

i.e.,

$$F(z) = \psi_m^*(z)/\phi_m^*(z)$$

and thus, by (1.14) and (1.11),

$$\mu'(\theta) = c_0 \operatorname{Re} F(e^{i\theta}) = 1/|\phi_m^*(e^{i\theta})|^2$$

and

$$D(e^{i\theta}, \mu') = 1/\phi_m^*(e^{i\theta}) \qquad \text{for} \quad \theta \in [-\pi, \pi],$$

we even have equality in (2.19) for each $n \geqslant m$ and not only in the limit. Indeed since $a_\nu = 0$ for $\nu = m, m+1, \ldots$ it follows that

$$\phi_{m+k}(z) = z^k \phi_m(z) \quad \text{and} \quad \phi^*_{m+k}(z) = \phi^*_m(z) \quad (k = 0, 1, 2, \ldots)$$
$$\psi_{m+k}(z) = z^k \psi_m(z) \quad \text{and} \quad \psi^*_{m+k}(z) = \psi^*_m(z) \quad (k = 0, 1, 2, \ldots)$$

and thus, using relations (1.17) and (1.18)

$$\tilde{g}_{m+k}(z) = \frac{2}{c_0} \frac{1}{\phi^*_m(z)} \quad \text{and} \quad \tilde{h}_{m+k}(z) = 0 \quad (k = 1, 2, \ldots).$$

Concerning the rate of convergence on the unit circle, which is of special interest in what follows, we get, with the help of Theorem 2.1, the following statement.

COROLLARY 2.1. *Let* $a \in (0, \pi]$ *and suppose that both limit relations in* (2.18) *hold uniformly for* $z = e^{i\theta}$, $\theta \in [-a, a]$, *where for* $n \in \mathbb{N}$

$$\phi^*_n(e^{i\theta}, d\mu) = D(e^{i\theta}, \mu')^{-1} + \varepsilon_n(e^{i\theta}) \tag{2.18'}$$

with, for $K_1 \in \mathbb{R}^+$,

$$|\varepsilon_n(e^{i\theta})| \leqslant K_1 \delta_n \qquad \text{for} \quad \theta \in [-a, a]$$

and

$$\lim_{n \to \infty} \delta_n = 0.$$

Then the following relationships hold for $n \in \mathbb{N}$:

$$\tilde{g}_n(e^{i\theta}, d\mu) = \frac{2}{c_0} D(e^{i\theta}, \mu') + \eta_{n,1}(e^{i\theta})$$

and

$$\tilde{h}_n(e^{i\theta}, d\mu) = \eta_{n,2}(e^{i\theta}),$$

with, for $K_2 \in \mathbb{R}^+$,

$$|\eta_{n,j}(e^{i\theta})| \leqslant K_2 \delta_n \qquad \text{for} \quad \theta \in [-a, a] \qquad (j = 1, 2).$$

Proof. From (2.18′) we get

$$\lim_{n \to \infty} \phi_n(e^{i\theta}) = e^{in\theta}/\overline{D(e^{i\theta})} + \overline{\tilde{\varepsilon}_n(e^{i\theta})} \tag{2.24}$$

with

$$|\tilde{\varepsilon}_n(e^{i\theta})| \leqslant K_1 \delta_n.$$

Hence it follows from relation (2.10) of Lemma 2.1 and (2.19) that

$$\phi_n^*(e^{i\theta})\ \tilde{g}_n(e^{i\theta}) = \frac{2}{c_0} + 0(\delta_n) \tag{2.25}$$

which, in conjunction with (2.18′), gives the assertion for $\tilde{g}_n$.

Concerning the rate of convergence of $\tilde{h}_n$ to zero the assertion follows immediately from relation (2.10), using (2.24) and (2.25). ∎

REMARK 2.1. *Let us state some conditions under which the assumptions of Theorem* 2.1(b) *are fulfilled*: (a) *If* $\mu' > 0$ *on* $[-\pi, \pi]$ *and* $\mu' \in C^1[-\pi, \pi]$ *or if* $\sum_{n=0}^{\infty} |a_n|$ *converges then both limit relations in* (2.19) *hold uniformly on* $|z| \leqslant 1$.

(b) *If* $\log \mu' \in L_1[-\pi, \pi]$ *and* $\mu' \in C^1[a, b]$, $[a, b] \subset [-\pi, \pi]$, *then both limit relations in* (2.19) *hold uniformly in* $|z| \leqslant r < 1$ *and on each subarc* $[e^{ia_1}, e^{ib_1}] \subset (e^{ia}, e^{ib})$.

(c) *Let* $R' > 1$. *If* $\mu'(\theta) = |f(e^{i\theta})|^2 \geq 0$ *for* $\theta \in [-\pi, \pi]$, *where* f *is analytic for* $|z| < R'$, *or if* $\limsup_{n \to \infty} \sqrt[n]{|a_n|} \leqslant 1/R'$, *then the limit relations in* (2.19) *hold uniformly for* $|z| \leqslant R < R'$.

Proof. (a) Applying Theorem 26.1 of [7] to $\{a_n\}$ and $\{-a_n\}$ it follows that $\sum_{n=0}^{\infty} |a_n| < \infty$ is sufficient for both limit relations in (2.19) to hold uniformly on $|z| \leqslant 1$. If $\mu' \in C^1[-\pi, \pi]$ then it is known (see [13, Exercise 3]) that the conjugate function $\overline{\mu'} \in \mathrm{Lip}_\lambda[-\pi, \pi]$ which implies that $F(e^{i\theta}, d\mu) = \mu'(\theta) + i\overline{\mu'}(\theta) \in \mathrm{Lip}_\lambda[-\pi, \pi]$ and thus, since $\mu' > 0$ on $[-\pi, \pi]$, we have by (1.15) that $(\mu^*)' \in \mathrm{Lip}_\lambda[-\pi, \pi]$. Hence, by (2.7) and (2.8) both limit relations in (2.19) hold uniformly on $[-\pi, \pi]$.

(b) can be proved analogously to the second assertion in Theorem 2.1(b).

(c) follows immediately from [9, p. 82]. See also [8, 20, 15]. ∎

REMARK 2.2. (a) *If* $\mu' > 0$ *on* $[-\pi, \pi]$ *and* $\mu' \in C^1[-\pi, \pi]$, *the error functions* $\eta_{n,j}$, $j = 1, 2$, *from Corollary* 2.1 *satisfy*

$$|\eta_{n,j}(e^{i\theta}, \mu')| \leqslant K_2 (\log n)/n \qquad \textit{for} \quad \theta \in [-\pi, \pi], \qquad j = 1, 2. \tag{2.26}$$

(b) *If* μ' *is of the form*

$$\mu'(\theta) = |f(e^{i\theta})|^2 \qquad \textit{for} \quad \theta \in [-\pi, \pi],$$

where f is analytic for $|z| < R$, $R > 1$, *then*

$$|\eta_{n,j}(e^{i\Theta}, \mu')| \leqslant K_3/R^{n+1} \qquad \text{for} \quad \theta \in [-\pi, \pi], j = 1, 2. \tag{2.27}$$

Proof. (a) follows immediately from Remark 2.1(a), Corollary 2.1, and relation (2.8).

(b) In view of [7, Theorem 26.1] and [9, p. 82] we get

$$|\varepsilon_n(e^{i\theta})| \leqslant K_3 \sum_{k=n}^{\infty} |a_n| \leqslant K_3 \sum_{k=n}^{\infty} R^{-k}$$

which together with Corollary 2.1 gives the assertion. ∎

3. Connection between Functions of the Second Kind on an Interval and on a Circle

In this section we show how to represent the nth function of the second kind q_n on the interval $[-1, 1]$ in terms of the nth function of the second kind $\tilde{g}_n$ and its associated function $\tilde{h}_n$ on the unit circle. With the help of this result we easily get the asymptotic behaviour of the function of the second kind outside of the interval $[-1, 1]$ as well as on the interval $[-1, 1]$.

Let us note that it follows immediately by the definition (1.35) of the nth function of the second kind that it is analytic on $\mathbb{C} \setminus [-1, 1]$ and at infinity and that it can be represented in the form

$$q_n(y, d\alpha) = p_n(y, d\alpha) \left(\frac{1}{s_0} \int_{-1}^{1} \frac{d\alpha(x)}{y - x} \right) - p_{n-1}^{(1)}(y, d\alpha). \tag{3.1}$$

THEOREM 3.1. *Let* α *be a positive measure on* $[-1, 1]$, *let* μ, F, s_0, A_n, B_n *be given by* (1.23), (1.25), (1.26), *and* (1.28), *respectively, and put* $l_0 = \int_{-1}^{1} (1 - x^2)\, d\alpha / \int_{-1}^{1} d\alpha$. *Then on writing* $y = \frac{1}{2}(z + z^{-1})$, *we have for* $|z| < 1$ *and for* $n \geqslant 2$

$$\begin{aligned} &l_0 B_n q_{n-1}(y, (1 - x^2)\, d\alpha) \\ &\quad = z^n \{ \tilde{g}_{2n-1}(z, d\mu) - \tilde{h}_{2n-1}(z, d\mu) \} \\ &\quad = z^{-n} \{ z(\phi_{2n-1} F + \psi_{2n-1})(z) - (\phi_{2n-1}^* F - \psi_{2n-1}^*)(z) \} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &A_n \sqrt{y^2 - 1}\, q_n(y, d\alpha) \\ &\quad = z^n \{ \tilde{g}_{2n-1}(z, d\mu) + \tilde{h}_{2n-1}(z, d\mu) \} \\ &\quad = z^{-n} \{ z(\phi_{2n-1} F + \psi_{2n-1})(z) + (\phi_{2n-1}^* F - \psi_{2n-1}^*)(z) \}. \end{aligned} \tag{3.3}$$

Proof. First let us demonstrate that relation (3.2) holds. From (1.35) we get with the help of the transformation $x = \cos\theta$ and the relation (1.31) that

$$l_0 B_n q_{n-1}(y, (1-x^2)\,d\alpha)$$
$$= \frac{z}{s_0}\int_0^\pi \frac{\begin{pmatrix} e^{-i(n-1)\theta}\phi_{2n-1}(e^{i\theta}) \\ \quad - e^{i(n-1)\theta}\overline{\phi_{2n-1}(e^{i\theta})})(e^{-i\theta}-e^{i\theta}) \end{pmatrix}}{(z^2 - 2z\cos\theta + 1)}\,d\mu$$
$$= \frac{z}{(1-z^2)}\frac{1}{2\pi c_0}\left\{\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(e^{-in\theta} - e^{-i(n-2)\theta}\right)\phi_{2n-1}(e^{i\theta})\,d\mu\right.$$
$$\left. - \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(e^{i(n-2)\theta} - e^{in\theta}\right)\overline{\phi_{2n-1}(e^{i\theta})}\,d\mu\right\},$$

where in order to get the second equality we have used the facts that

$$\frac{1-z^2}{1+z^2-2\cos\theta z} = \frac{e^{i\theta}+z}{e^{i\theta}-z} + \frac{e^{-i\theta}+z}{e^{-i\theta}-z},$$

that μ is symmetric, and that $s_0 = \pi c_0$.

Using the fact that, by (2.14), for $m = 0, 1, \ldots, n-1$,

$$\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} e^{-im\theta}\phi_n(e^{i\theta})\,d\mu = z^{-m}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\,d\mu$$

and, which can be demonstrated analogously,

$$\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} e^{im\theta}\overline{\phi_n(e^{i\theta})}\,d\mu = z^m\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\overline{\phi_n(e^{i\theta})}\,d\mu,$$

relation (3.2) follows by straightforward calculation.

Relation (3.3) can be demonstrated analogously or directly using the relations (3.1), (1.25), (1.29), and (1.32). ∎

From Theorem 3.1 it follows that $z^{-n+1}g_{2n-1}$, $z^{-n}h_{2n-1}$, $z^{-n}g_{2n}$, and $z^{-n}h_{2n}$ can be represented as linear combinations of $q_{n-1}(y, (1-x^2)\,d\alpha)$ and $\sqrt{y^2-1}\,q_n(y, d\alpha)$, $y = \frac{1}{2}(z+z^{-1})$, where the representations of $z^{-n+1}g_{2n-1}$ and $z^{-n}h_{2n-1}$ follow immediately from (3.2) and (3.3) while the representation for $z^{-n}g_{2n}$ and $z^{-n}h_{2n}$ follow from the second equality in (3.2) and (3.3) in conjunction with the relations

$$\phi_{2n} \underset{(-)}{+} \phi_{2n}^* = (1 \underset{(+)}{-} a_{2n-1})(z\phi_{2n-1} \underset{(-)}{+} \phi_{2n-1}^*),$$
$$\psi_{2n} \underset{(-)}{+} \psi_{2n}^* = (1 \underset{(-)}{+} a_{2n-1})(z\psi_{2n-1} \underset{(-)}{+} \psi_{2n-1}^*),$$

which can easily be deduced from (1.2) and (1.2′) resp. from (1.9).

As a byproduct we obtain from Theorem 3.1 the following interesting representation of $p_n(x, d\alpha^*)$.

COROLLARY 3.1. *Let α^* be the measure associated with α by relation* (1.27). *Then*

$$p_n(y, d\alpha^*)=\frac{1}{s_0}\int_{-1}^{1}\frac{\begin{pmatrix}(y^2-1)\,p_{n-1}(y,(1-x^2)\,d\alpha)\\ \quad -(x^2-1)\,p_{n-1}(x,(1-x^2)\,d\alpha)\end{pmatrix}}{y-x}\,d\alpha(x).$$

Proof. Let l_0 be defined as in Theorem 3.1. On the one hand we have in view of (3.2), (1.30), (1.31), and the relation

$$(z^{-1}-z)/2=\sqrt{y^2-1} \qquad \text{for} \quad |z|<1$$

that

$$\begin{aligned} &l_0 q_{n-1}(y,(1-x^2)\,d\alpha)\\ &\quad = p_n(y, d\alpha^*)-\sqrt{y^2-1}\,p_{n-1}(y,(1-x^2)\,d\alpha)\,F(z, d\mu), \end{aligned}$$

and on the other hand we get from (1.35)

$$\begin{aligned} &l_0 q_{n-1}(y,(1-x^2)\,d\alpha)\\ &\quad =\frac{1}{s_0}\int_{-1}^{1}\frac{\begin{pmatrix}(y^2-1)\,p_{n-1}(y,(1-x^2)\,d\alpha)\\ \quad -(x^2-1)\,p_{n-1}(x,(1-x^2)\,d\alpha)\end{pmatrix}}{y-x}\,d\alpha\\ &\qquad -(y^2-1)\,p_{n-1}(y,(1-x^2)\,d\alpha)\left(\frac{1}{s_0}\int_{-1}^{1}\frac{d\alpha(x)}{y-x}\right), \end{aligned}$$

which in view of (1.25) gives the assertion. ∎

Next let us consider the nth function of the second kind for weight functions of Bernstein–Szegö type, i.e., for weight functions of the form

$$W(x)=1/(\sqrt{1-x^2}\,\rho_m(x)) \qquad \text{for} \quad x\in(-1,1), \tag{3.4}$$

where ρ_m is a polynomial of degree m which is positive on $[-1,1]$. It is well known that ρ_m has a unique representation of the form

$$\rho_m(x)=c\,|\Phi_m^*(e^{i\theta})|^2, \qquad x=\cos\theta,\ \theta\in[0,\pi],\ c\in\mathbb{R}_+, \tag{3.5}$$

where

$$\Phi_m(z)=\prod_{\nu=1}^{m}(z-z_\nu) \text{ with } |z_\nu|<1 \qquad \text{for} \quad \nu=1,\ldots,m,$$

and the z_ν are real or appear in pairs of complex conjugate numbers. For such weight functions we obtain from Theorem 3.1 the following simple explicit expression for the nth function of the second kind which we have derived by different methods in [25, (2.19)].

COROLLARY 3.2. *Let ρ_m and W be given by* (3.4) *and* (3.5), *respectively. Then, on writing $y=\frac{1}{2}(z+z^{-1})$, $|z|<1$, we have for $y\in\mathbb{C}\setminus[-1,1]$ and for $2n\geqslant m+1$*

$$\sqrt{2\pi}\sqrt{y^2-1}\,q_n(y,W)=l_0\sqrt{2\pi}\,q_{n-1}(y,(1-x^2)W)$$
$$=\frac{2}{c_0\sqrt{c}}\frac{z^n}{\Phi_m^*(z)}.$$

Proof. Easy calculation gives, using Cauchy's Theorem, that $\phi_m:=\sqrt{c}\,\Phi_m$ is orthonormal with respect to $\mu'(\theta):=W(\cos\theta)\,|\sin\theta|=1/|\phi_m^*(e^{i\theta})|^2$ and thus the relations given in the example following Theorem 2.1 can be applied, which by Theorem 3.1 gives the assertion. ∎

With the help of Theorem 2.1 and 3.1 we obtain the following asymptotic representation of the function of the second kind outside of the interval $[-1,1]$ due to Barrett [2] and Geronimus [11, (XII.34)], where the last-named author gave a very short and elegant proof.

THEOREM 3.2. (Barrett [2], Geronimus [11]). *If the weight function w is in the Szegö class, i.e., if $(1-x^2)^{-1/2}\log w(x)\in L^1[-1,1]$, then the relation*

$$\frac{\sqrt{y^2-1}\,q_n(y,w)}{(y-\sqrt{y^2-1})^n}=\frac{\sqrt{2\pi}}{s_0}D(y-\sqrt{y^2-1},w(\cos\theta)\,|\sin\theta|)+\varepsilon_n(y,w)$$

holds, where $\lim_{n\to\infty}\varepsilon_n(y,w)=0$ uniformly on each compact subset of $\mathbb{C}\setminus[-1,1]$.

Proof. Let A_n be defined as in (1.28). In view of relation (3.3) we have for $|z|<1$ and $y=\frac{1}{2}(z+z^{-1})$

$$z^{-n}A_n\sqrt{y^2-1}\,q_n(y,w)=\tilde{g}_{2n-1}(z,d\mu)+\tilde{h}_{2n-1}(z,d\mu),$$

where $\mu'(\theta)=w(\cos\theta)\,|\sin\theta|$. Taking into consideration the facts that

$$z=y-\sqrt{y^2-1}=1/(y+\sqrt{y^2-1})$$

and that by (2.5)

$$\lim_{n\to\infty}A_n=\sqrt{2\pi},$$

the assertion follows immediately from Theorem 2.1 since, as mentioned at the beginning of Section 2, relation (2.2) holds. ▌

In order to study the asymptotic behavior of functions of the second kind on $[-1, 1]$ let us put, for $x \in [-1, 1]$,

$$q_n^+(x, d\alpha) := \lim_{\varepsilon \to 0^+} q_n(x + i\varepsilon, d\alpha)$$

and

$$q_n^-(x, d\alpha) := \lim_{\varepsilon \to 0^+} q_n(x - i\varepsilon, d\alpha).$$

Hence if q_n^+ and q_n^- exist we have

$$q_n(x, d\alpha) = \frac{q_n^+(x, d\alpha) + q_n^-(x, d\alpha)}{2} \qquad \text{for} \quad x \in [-1, 1].$$

THEOREM 3.3. *Let* $l_1 = \sqrt{2\pi}/\int_{-1}^{1} (1 - x^2)\, w(x)\, dx$, $l_2 = -\sqrt{2\pi}/s_0$, *and* $\mu'(\theta) = w(\cos\theta)\, |\sin\theta|$. *The asymptotic representations*

$$q_{n-1}^{\overset{-}{(+)}}(\cos\theta, (1 - x^2) w) = l_1 e^{\overset{+}{(-)} in\theta} D(e^{\overset{+}{(-)} i\theta}, \mu') + \varepsilon_{n,1}(e^{i\theta}) \tag{3.6}$$

$$q_{n-1}(\cos\theta, (1 - x^2) w) = l_1 \operatorname{Re}\{e^{in\theta} D(e^{i\theta}, \mu') + \varepsilon_{n,1}(e^{i\theta})\} \tag{3.7}$$

$$\sin\theta\, q_n(\cos\theta, w) = l_2 \operatorname{Im}\{e^{in\theta} D(e^{i\theta}, \mu') + \varepsilon_{n,2}(e^{i\theta})\} \tag{3.8}$$

hold with

$$\lim_{n \to \infty} \varepsilon_{n,j}(e^{i\theta}) = 0 \qquad (j = 1, 2)$$

(a) *uniformly for* $\theta \in [0, 2\pi]$ *if* $\sqrt{1 - x^2}\, w(x) \geqslant m > 0$ *and* $\sqrt{1 - x^2}\, w(x) \in C^1[-1, 1]$,

(b) *uniformly for* $\theta \in [\arccos \xi_2 + \delta, \arccos \xi_1 - \delta]$, $\delta > 0$, *if* $0 < m \leqslant \sqrt{1 - x^2}\, w(x) \leqslant M$ *and* $\sqrt{1 - x^2}\, w(x) \in C^1[\xi_1, \xi_2]$, $[\xi_1, \xi_2] \subset [-1, +1]$.

More precisely, if

$$|\phi_n^*(e^{i\theta}, d\mu) - D(e^{i\theta}, d\mu)^{-1}| \leqslant K_1 \delta_n \qquad \textit{for} \quad n \in \mathbb{N}$$

on the set where uniform convergence occurs in (a) *or* (b), *then*

$$|\varepsilon_{n,j}(e^{i\theta})| \leqslant K_2 \delta_n \qquad \textit{for} \quad n \in \mathbb{N} \qquad (j = 1, 2)$$

on this set.

Proof. Using the fact that

$$(\sqrt{y^2-1})^+ (x) = -(\sqrt{y^2-1})^- (x) = i\sqrt{1-x^2} \qquad \text{for} \quad x \in [-1, 1],$$

where that branch of $\sqrt{y^2-1}$ which is positive on $(1, \infty)$ is chosen, we have

$$(y-\sqrt{y^2-1})^+ (\cos\theta) = e^{-i\theta}$$

and

$$(y-\sqrt{y^2-1})^- (\cos\theta) = e^{i\theta} \qquad \text{for} \quad \theta \in [0, \pi].$$

Since in Theorem 3.1 $y \in \mathbb{C} \backslash [-1, 1]$ and $z \in \{z \in \mathbb{C} : |z| < 1\}$ are related to each other by $z = y - \sqrt{y^2-1}$ we get from Theorem 3.1, observing that the boundary values $\lim_{z \to e^{i\theta}} F(z, d\mu) = \lim_{z \to e^{i\theta}} \sqrt{y^2-1}\, q_0(y, w)$, by the assumptions on $\mu'(\theta) = w(\cos\theta)\, |\sin\theta|$, certainly exist,

$$\begin{aligned} &l_0 B_n q_{n-1}^{\overset{-}{(+)}}(\cos\theta, (1-x^2)w) \\ &\quad = e^{\overset{+}{(-)} in\theta}\{\tilde{g}_{2n-1}(e^{\overset{+}{(-)} i\theta}, d\mu) - \tilde{h}_{2n-1}(e^{\overset{+}{(-)} i\theta}, d\mu)\} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &A_n \sin\theta q_n^{\overset{-}{(+)}}(\cos\theta, w) \\ &\quad = \underset{(-)}{+} i e^{\overset{+}{(-)} in\theta}\{\tilde{g}_{2n-1}(e^{\overset{+}{(-)} i\theta}, d\mu) + \tilde{h}_{2n-1}(e^{\overset{+}{(-)} i\theta}, d\mu)\}. \end{aligned} \tag{3.10}$$

Recalling that

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} B_n = \sqrt{2\pi} \tag{3.11}$$

the assertion now follows from Theorem 2.1 combined with Remark 2.1, Corollary 2.1, and Remark 2.2. ∎

Example. Let us consider the Jacobi weight function $w(x) = (1-x)^{\alpha-1/2}(1+x)^{\beta-1/2}$ on $[-1, 1]$. We get immediately

$$\mu'(\theta) = (1-\cos\theta)^\alpha (1+\cos\theta)^\beta$$

and

$$D(z, \mu') = 2^{-(\alpha+\beta)/2}(1-z)^\alpha (1+z)^\beta.$$

In view of Theorem 3.3 we get the known asymptotic formula

$$
\begin{aligned}
q_n(\cos\theta, w) + \varepsilon_n(\cos\theta)
&= -\frac{\sqrt{2\pi}}{s_0} 2^{-(\alpha+\beta)/2} \frac{\operatorname{Im}\{e^{in\theta}(1-e^{i\theta})^{\alpha}(1+e^{i\theta})^{\beta}\}}{\sin\theta} \\
&= -\frac{\sqrt{2\pi}}{s_0} (1-\cos\theta)^{(\alpha-1)/2} (1+\cos\theta)^{(\beta-1)/2} \\
&\quad \times \sin((n+(\alpha+\beta)/2)\theta - \alpha\pi/2),
\end{aligned}
$$

where $\lim_{n\to\infty} \varepsilon_n(\cos\theta) = 0$ uniformly on $[\delta, \pi-\delta]$, $\delta > 0$.

It is well known that the Chebyshev polynomials of the first and second kinds are the solutions of a certain singular integral equation. From Theorem 3.3 we obtain the interesting fact that a corresponding result holds asymptotically for the polynomials $p_n(x, w)$ and $p_{n-1}(x, (1-x^2)w)$.

COROLLARY 3.3. *The relations hold, for* $y \in (-1, 1)$,

$$
\begin{aligned}
&\frac{1}{\pi} \int_{-1}^{1} \frac{p_n(x, w)}{y-x} w(x)\, dx \\
&\quad = -w(y)\sqrt{1-y^2}\, p_{n-1}(y, (1-x^2)w) + \tilde{\varepsilon}_{n,2}(y)
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\frac{1}{\pi} \int_{-1}^{1} \frac{p_{n-1}(x, (1-x^2)w)}{y-x} (1-x^2)\, w(x)\, dx \\
&\quad = w(y)\sqrt{1-y^2}\, p_n(y, w) + \tilde{\varepsilon}_{n,1}(y),
\end{aligned}
\tag{3.13}
$$

where $\tilde{\varepsilon}_{n,j}(\cos\theta)$, $j \in \{1, 2\}$, *converges uniformly on the same set and at least with the same rate as the function* $\varepsilon_{n,j}(e^{i\theta})$, $j \in \{1, 2\}$, *from Theorem* 3.3.

Proof. Obviously we have

$$
\operatorname{Re}\{e^{in\theta} D(e^{i\theta}, \mu')\} = |D(e^{i\theta}, \mu')|^2 \operatorname{Re}\{e^{in\theta} D(e^{-i\theta}, \mu')\}
$$

and a corresponding relation holds for the imaginary part. Taking into account that by (1.29), respectively (1.31), (2.7) and (3.11), $y = \cos\theta$,

$$
2\operatorname{Re}\{e^{in\theta} D(e^{-i\theta}, \mu')\} = \sqrt{2\pi}\, p_n(y, w) + \tilde{\varepsilon}_{n,1}(y),
$$

respectively

$$\frac{2\,\mathrm{Im}\{e^{in\theta}D(e^{-i\theta},\mu')\}}{\sin\theta}=\sqrt{2\pi}\,p_{n-1}(y,(1-x^2)w)+\tilde{\varepsilon}_{n,2}(y),$$

the assertion follows immediately on recalling Definition (1.37). ∎

If the weight function is of Bernstein–Szegö type then again we even have equality in (3.12) and (3.13) and not only in the limit.

COROLLARY 3.4. *Let ρ_m be polynomial of exact degree m, which is positive on $[-1, 1]$, and set*

$$W(x)=1/\sqrt{1-x^2}\,\rho_m(x) \qquad \text{for} \quad x\in(-1,1).$$

Then the following relations hold for $2n-1\geq m$:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p_n(x,W)}{y-x}\frac{1}{\rho_m(x)\sqrt{1-x^2}}\,dx=\frac{-1}{\rho_m(y)}\,p_{n-1}(y,(1-x^2)W)$$

and

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p_{n-1}(x,(1-x^2)W)}{y-x}\frac{\sqrt{1-x^2}}{\rho_m(x)}\,dx=\frac{1}{\rho_m(y)}\,p_n(y,W).$$

Proof. Let Φ_m be defined by (3.5). Since, as already mentioned, $\phi_m:=\sqrt{c}\,\Phi_m$ is orthonormal on $|z|=1$ with respect to $\mu'(\theta):=W(\cos\theta)\,|\sin\theta|=1/|\phi_m^*(e^{i\theta})|^2$, the results of the example following Theorem 2.1 can be applied. Hence for $2n-1\geq m$

$$\tilde{g}_{2n-1}(z,\mu')=\frac{2\pi}{s_0}\frac{1}{\phi_m^*(z)} \qquad \text{and} \qquad \tilde{h}_{2n-1}(z)=0,$$

and with the help of (1.29) and (1.31), $y=\cos\theta$,

$$\sqrt{2\pi}\,p_n(y,W)=2\,\mathrm{Re}\{e^{i(n-m)\theta}\phi_m(e^{i\theta})\}$$

and

$$\sqrt{2\pi}\,p_{n-1}(y,(1-x^2)W)=2\,\mathrm{Im}\{e^{i(n-m)\theta}\phi_m(e^{i\theta})\}/\sin\theta.$$

The assertion now follows from (3.9) and (3.10) by simple calculation.

Corollary 3.4 could also be proved with the help of Corollary 3.2. ∎

Setting $\rho_m\equiv 1$ in Corollary 3.4 we obtain the well known integral equations for the Chebyshev polynomials of first, respectively second kind.

4. On Stieltjes and Geronimus Polynomials

In this section we first derive asymptotic formulas for Stieltjes and Geronimus polynomials defined in (1.38) and (1.40). With the help of these results for a quite general class of weight functions it is shown that all zeros of the Stieltjes and Geronimus polynomials are simple and contained in $(-1, 1)$ and that the Kronrod quadrature formula has all quadrature weights positive.

The next lemma, part (a) of which we have given in a slightly different form in [24, Lemma 3] and part (b) of which is essentially contained in [18, Theorem 2], shows how Stieltjes, respectively Geronimus polynomials are related to functions of the second kind on the unit circle.

Lemma 4.1. *Let*

$$Q_n(z, d\alpha) := \frac{2^{n+1}}{k_n(d\alpha)} \frac{z^{n+1}}{s_0 q_n(\frac{1}{2}(z+z^{-1}), d\alpha)}$$

$$= 1 + \sum_{j=1}^{\infty} d_{j,n} z^j \qquad \text{for} \quad |z| < 1. \tag{4.1}$$

(a) *The Stieltjes polynomial* $E_{n+1}(\cdot, d\alpha)$ *is given by*

$$E_{n+1}(\cos\theta, d\alpha) = 2^{-n} \operatorname{Re}\{S^*_{n+1}(e^{i\theta}, Q_n(\cdot, d\alpha)) - d_{n+1,n}/2\};$$

(b) *the Geronimus polynomial* $J_n(\cdot, d\alpha)$ *by*

$$J_n(\cos\theta, d\alpha) = 2^{-n} \operatorname{Im}\{S^*_{n+1}(e^{i\theta}, Q_n(\cdot, d\alpha))\},$$

where $S_n(z, f)$ *denotes the nth partial sum of the series expansion of* f *at* $z=0$.

Proof. (b) Let

$$2^n J_n(y, d\alpha) = \sum_{j=0}^{n} b_j U_j(y)$$

$$= \sum_{j=0}^{n} b_j \left(\frac{(y+\sqrt{y^2-1})^{j+1} - (y-\sqrt{y^2-1})^{j+1}}{2\sqrt{y^2-1}} \right).$$

If we set $z = y - \sqrt{y^2-1}$, $y = \frac{1}{2}(z+z^{-1})$, then we obtain from (1.40) in a neighbourhood of the point zero that

$$\frac{(2z)^{n+1}}{k_n s_0 q_n(\frac{1}{2}(z+z^{-1}), d\alpha)} = \sum_{j=0}^{n} b_j z^{n-j} + 0(z^{n+1})$$

and thus

$$S_n^*(z, Q_n(\cdot, d\alpha)) = \sum_{j=0}^{n} b_{n-j} z^{n-j}$$

which immediately implies

$$S_{n+1}^*(z, Q_n(\cdot, d\alpha)) = \sum_{j=0}^{n} b_{n-j} z^{n+1-j} + d_{n+1,n},$$

which proves part (b). ∎

Now let us give the announced asymptotic formula for Stieltjes and Geronimus polynomials.

THEOREM 4.1. (a) *Suppose that the weight function w satisfies*

$$0 < m \leqslant \sqrt{1-x^2}\, w(x) \qquad \textit{for} \quad x \in [-1, 1]$$

and (4.2)

$$\sqrt{1-x^2}\, w(x) \in C^2[-1, 1].$$

Let $\mu'(\theta) = w(\cos\theta)\, |\sin\theta|$ *and* $\hat{D}(\cdot, \mu') = D(\cdot, \mu')/D(0, \mu')$. *Then the Stieltjes, respectively Geronimus polynomials have the asymptotic representation*

$$2^n E_{n+1}(\cos\theta, (1-x^2)w) = \operatorname{Re}\{S_{n+1}^*(e^{i\theta}, \hat{D}(\cdot, \mu')^{-1}) + \varepsilon_n(e^{i\theta}, \mu')\}, \tag{4.3}$$

respectively

$$2^n J_n(\cos\theta, (1-x^2)w) = \operatorname{Im}\{S_{n+1}^*(e^{i\theta}, \hat{D}(\cdot, \mu')^{-1}) + \varepsilon_n(e^{i\theta}, \mu')\} \tag{4.4}$$

where $\lim_{n\to\infty} \varepsilon_n(e^{i\theta}, \mu') = 0$ *uniformly on* $[-\pi, \pi]$. *More precisely, for* $n \in \mathbb{N}$,

$$|\varepsilon_n(e^{i\theta}, \mu')| < C(\log n)/n.$$

(b) *If w satisfies the conditions*

$$w(x)/\sqrt{1-x^2} \in L^1[-1, 1], \sqrt{1-x^2}\, w(x) \geqslant m > 0$$
$$\textit{for} \quad x \in [\xi_1, \xi_2] \subset [-1, 1]$$

and (4.5)

$$\sqrt{1-x^2}\, w(x) \in C^2[\xi_1, \xi_2]$$

then

$$2^n E_{n+1}(\cos\theta, (1-x^2)w) \\ = \operatorname{Re}\{S^*_{n+1}(e^{i\theta}, \hat{D}(\cdot, \mu')^{-1}) - (d_{n+1,n}/2) + \varepsilon_n(e^{i\theta}, \mu')\},$$

where $\lim_{n\to\infty} \varepsilon_n(e^{i\theta}, \mu') = 0$ *uniformly for* $\theta \in [\arccos \xi_2 + \eta, \arccos \xi_1 - \eta]$, $\eta > 0$. *Furthermore relation* (4.4) *holds uniformly on* $[\arccos \xi_2 + \eta,$ $\arccos \xi_1 - \eta]$, $\eta > 0$.

Proof. Concerning part (a) put

$$Q_n(z) := Q_n(z, (1-x^2)w),$$
$$k_n := k_n((1-x^2)\,d\alpha)), \qquad \text{and} \qquad D(z) := D(z, \mu')$$

where $Q_n(z, (1-x^2)w)$ is defined in (4.1). From Theorem 3.1 combined with Theorem 2.1 and Remark 2.1 we get

$$1 = \lim_{n\to\infty} Q_n(0) = \lim_{n\to\infty} \frac{2^{n+1}}{\sqrt{2\pi}\, k_n D(0, \mu')}$$

where we have taken into account (4.1), and

$$Q_n(e^{i\theta}) := \lim_{r\to 1^-} Q_n(re^{i\theta}) = \hat{D}(e^{i\theta})^{-1} + \varepsilon_n(e^{i\theta}) \tag{4.6}$$

where, by Remark 2.2,

$$|\varepsilon_n(e^{i\theta})| \leqslant K_1(\log n)/n \qquad \text{for} \quad \theta \in [-\pi, \pi].$$

Since $\mu' \in C^2[-\pi, \pi]$ implies (see, e.g., [13, Exercise 3]) that the conjugate function $\overline{\mu'} \in C^1[-\pi, \pi]$, the function $F := F(\cdot, d\mu)$ defined in (1.12) has the properties, see (1.14),

$$c_0 F(e^{i\theta}) = \mu'(\theta) + i\overline{\mu'}(\theta) \qquad \text{and} \qquad F(e^{i\theta}) \in C^1[-\pi, \pi].$$

Thus it follows with the help of the second relation in (1.17) and (1.18) that $\tilde{g}_{2n+1}(e^{i\theta}, d\mu) - \tilde{h}_{2n+1}(e^{i\theta}, d\mu) \in C^1[-\pi, \pi]$. Using the fact that by (1.22′) $D(e^{i\theta}) \neq 0$ for $\theta \in [-\pi, \pi]$, since $\mu' > 0$, and thus, by (4.6), $Q_n(e^{i\theta}) \neq 0$ on $[-\pi, \pi]$ for $n \geqslant n_0$, we get that $Q_n(e^{i\theta}) \in C^1[-\pi, \pi]$ and moreover satisfies a Lipschitz condition on $[-\pi, \pi]$ with Lip-constant L_n for $n \geqslant n_0$. Observing that by assumption on μ', $\log \mu' \in C^2[-\pi, \pi]$ and thus the conjugate function $\overline{\log \mu'} \in C^1[-\pi, \pi]$ we obtain with the help of (1.20) that $\log D(e^{i\theta}) \in C^1[-\pi, \pi]$ and thus $\hat{D}(e^{i\theta})^{-1} \in C^1[-\pi, \pi]$. Hence $\hat{D}^{-1}$ satisfies a Lip-condition on $[-\pi, \pi]$ which implies on the one hand by well known results on Fourier series (see, e.g., [30, Chap. II]) that

$$|\hat{D}(e^{i\theta})^{-1} - S_{n+1}(e^{i\theta}, \hat{D}^{-1})| \leqslant K_2(\log n)/n \qquad \text{for} \quad \theta \in [-\pi, \pi], \tag{4.7}$$

and on the other hand, in view of the uniform convergence of Q_n to $\hat{D}^{-1}$, that the Lip-constants L_n of Q_n, $n \geqslant n_0$, are bounded by a constant K_3. Hence

$$|Q_n(e^{i\theta}) - S_{n+1}(e^{i\theta}, Q_n)| \leqslant K_3(\log n)/n \quad \text{for} \quad \theta \in [-\pi, \pi] \text{ and } n \geqslant n_0, \tag{4.8}$$

which, in conjunction with (4.6) and (4.7), gives

$$|S_{n+1}(e^{i\theta}, \hat{D}^{-1}) - S_{n+1}(e^{i\theta}, Q_n)| \leqslant K_4(\log n)/n \quad \text{for} \quad \theta \in [-\pi, \pi] \text{ and } n \geqslant n_0. \tag{4.9}$$

Next let us demonstrate that

$$\lim_{n \to \infty} d_{n+1,n} = 0. \tag{4.10}$$

If we set

$$\hat{D}^{-1}(z) = \sum_{n=0}^{\infty} d_n z^n$$

then we have, since $\log \mu' \in L^1[-\pi, \pi]$, that $\hat{D}^{-1} \in H_2$ and thus

$$\lim_{n \to \infty} d_n = 0.$$

Since $\operatorname{Re} Q_n(e^{i\theta})$ and $\operatorname{Re} \hat{D}(e^{i\theta})^{-1}$ are continuous on $[-\pi, \pi]$ we get by applying Schwarz's formula (see, e.g., [30]) and by taking into consideration relation (4.6) that

$$|d_{n+1,n} - d_n| = \left| \frac{1}{\pi i} \int_{|z|=1} \operatorname{Re}\{Q_n(z) - \hat{D}(z)^{-1}\} \frac{dz}{z^{n+2}} \right| \leqslant K_5 \max_{\theta \in [-\pi,\pi]} |\varepsilon_n(e^{i\theta})| \xrightarrow[n \to \infty]{} 0,$$

which proves (4.10). Part (a) now follows from (4.9) and Lemma 4.1.

Concerning part (b) it is not difficult to check that all arguments used to prove relation (4.9) also hold locally, where in (4.6) and (4.9) $(\log n)/n$ is to be replaced by δ_n where $\delta_n \xrightarrow[n \to \infty]{} 0$. ∎

As a consequence of Theorem 4.1 we obtain the following important asymptotic relations.

COROLLARY 4.1. *Let k_n denote the leading coefficient of $p_n(x, (1-x^2)w)$.*

(a) *Suppose that the weight function w satisfies the assumption of Theorem* 4.1(a). *Then on* $[-1, 1]$ *the asymptotic relationship holds that*

$$k_n E_{n+1}(x, (1-x^2)w) = p_{n+1}(x, w) + \delta_{n,1}(x), \tag{4.11}$$

and

$$k_n J_n(x, (1-x^2)w) = p_n(x, (1-x^2)w) + \delta_{n,2}(x), \tag{4.12}$$

where

$$|\delta_{n,j}(x)| \leqslant C(\log n)/n \qquad \text{for} \quad x \in [-1, 1], n \in \mathbb{N} \text{ and } j = 1, 2.$$

(b) *If w satisfies the assumptions of Theorem* 4.1(b) *then*

$$k_n E_{n+1}(x, (1-x^2)w) + k_n d_{n+1,n}/2^{n+1} = p_{n+1}(x, w) + \tilde{\delta}_{n,1}(x) \tag{4.13}$$

and

$$k_n J_n(x, (1-x^2)w) = p_n(x, (1-x^2)w) + \tilde{\delta}_{n,2}(x), \tag{4.14}$$

where $\lim_{n \to \infty} \tilde{\delta}_{n,j}(x) = 0$ *uniformly on* $[\xi_1 + \delta, \xi_2 - \delta]$, $\delta > 0$.

Proof. (a) First let us prove (4.11). In view of (1.29) and (2.7) we have

$$A_n p_n(\cos\theta, w) = 2 \operatorname{Re}\{e^{-in\theta} D(e^{i\theta}, \mu')^{-1} + \varepsilon_n(e^{i\theta})\}, \tag{4.15}$$

where by (2.8)

$$|\varepsilon_n(e^{i\theta})| \leqslant K_1(\log n)/n \qquad \text{for} \quad \theta \in [-\pi, \pi].$$

Thus we get with the help of (4.3) and (4.7), recalling (1.1) and (2.2), that

$$\kappa 2^{n+1} E_{n+1}(\cos\theta, (1-x^2)w) = A_{n+1} p_{n+1}(x, w) + \delta_n(x) \tag{4.16}$$

with

$$|\delta_n(x)| \leqslant K_2(\log n)/n \qquad \text{for} \quad x \in [-1, 1].$$

Next we claim that

$$k_n^2 = 2^{2n+1}(1 + a_{2n+1}(d\mu))\, \kappa_{2n+2}^2(d\mu)/\pi \tag{4.17}$$

and

$$|a_n(d\mu)| \leqslant K_3(\log n)/n. \tag{4.18}$$

Indeed, (4.17) follows immediately from the fact that

$$\int_{-1}^{1} p_n^2(x, (1-x^2)w)(1-x^2)w(x)\, dx = \pi/2^{2n+1}(1 + a_{2n+1}(d\mu))\kappa_{2n+2}^2(d\mu),$$

which can be demonstrated as in [7, (31.12)] by using relation (1.31). Concerning relation (4.18) let us recall that, see [10, (1.9)],

$$\frac{\kappa_n}{\kappa}\phi_n^*(z, d\mu) = \frac{1}{\kappa}\sum_{k=0}^{n} \overline{\phi_k(0, d\mu)}\,\phi_k(z, d\mu)$$

and hence

$$D(e^{i\theta}, \mu')^{-1} = \frac{1}{\kappa}\sum_{k=0}^{\infty} \overline{\phi_k(0, d\mu)}\,\phi_k(z, d\mu),$$

which implies that

$$\begin{aligned}|a_n| &\leqslant \sqrt{\sum_{k=n+1}^{\infty} \frac{|\phi_k(0)|^2}{\kappa^2}}\\ &= \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |D(e^{i\theta})^{-1} - \frac{\kappa_n}{\kappa}\phi_n^*(e^{i\theta})|^2\, d\theta\right\}^{1/2} \leqslant K_3 \frac{\log n}{n},\end{aligned}$$

where the first inequality follows by relation (1.4) and the fact that $\kappa_0 \leqslant \kappa_1 \leqslant \cdots \leqslant \kappa_n \leqslant \cdots \leqslant \kappa$, and the last inequality follows by (2.8) and the estimate $|\kappa - \kappa_n| \leqslant K(\log n)/n$ which can be deduced from (2.2) and (2.8) combined with the maximum principle. Hence (4.18) is proved.

Relation (4.11) now follows from relation (4.16) together with (4.17) and (4.18) if we take into account that by (4.15) $\{p_n(x, w)\}$ is uniformly bounded on $[-1, 1]$. Along the same lines we obtain (4.12).

Part (b) can be demonstrated analogously. ∎

Concerning the asymptotic behaviour on $\mathbb{C}\setminus[-1, 1]$ we get

COROLLARY 4.2. *Suppose that w satisfies the assumptions* (4.2) *and set $\mu'(\theta) = w(\cos\theta)\,|\sin\theta|$ for $\theta \in [-\pi, \pi]$. Then*

$$\lim_{n\to\infty} \frac{2^{n+1}E_{n+1}(y, (1-x^2)w)}{(y+\sqrt{y^2-1})^{n+1}} = \hat{D}(y - \sqrt{y^2-1}, \mu')^{-1} \tag{4.19}$$

holds in the domain $|y + \sqrt{y^2-1}| > 1$, the converge being uniform for $|y+\sqrt{y^2-1}| \geqslant 1+\eta$, $\eta > 0$.

Proof. In view of Lemma 4.1, relation (4.9), and the maximum principle we have, for $z = y - \sqrt{y^2-1} = 1/(y+\sqrt{y^2-1})$,

$$\begin{aligned}(2z)^{n+1}\, E_{n+1}(y) &= z^{n+1}S_{n+1}^*(z, Q_n) + S_{n+1}(z, Q_n)\\ &= z^{n+1}S_{n+1}^*(z, \hat{D}^{-1}) + S_{n+1}(z, \hat{D}^{-1}) + \tilde{\varepsilon}_n(z)\end{aligned}$$

with $\lim_{n\to\infty} \tilde{\varepsilon}_n(z)=0$ for $|z|\leqslant 1$. Since relation (4.7) implies that $S_{n+1}(z,\hat{D}^{-1})$ converges to $\hat{D}^{-1}$ on $|z|\leqslant 1$ and that $S^*_{n+1}(z,\hat{D}^{-1})$ is bounded on $|z|=1$ the assertion is proved. ∎

It would be interesting to know whether the limit relation (4.19) holds under weaker conditions on the weight function w.

Based on numerical results Monegato conjectured in [18] that the zeros of the Stieltjes polynomials E_{n+1} and E_n for the Legendre weight separate each other. Among other results we show in the next corollary that this is true, at least for large n, for weight functions $(1-x^2)w$ satisfying assumption (4.2). For the Legendre weight this question remains still open.

COROLLARY 4.3. (a) *Suppose that* w *satisfies the assumption* (4.2) *of Theorem* 4.1(a). *Then there exists an* n_0 *such that for* $n\geqslant n_0$, $E_n(\cdot,(1-x^2)w)$ *and* $J_n(\cdot,(1-x^2)w)$ *have n simple zeros in* $(-1,1)$ *and the zeros of* $(x^2-1)\,p_n(\cdot,(1-x^2)w)$ *and* $E_{n+1}(\cdot,(1-x^2)w)$, $p_{n+1}(\cdot,w)$ *and* $J_n(\cdot,(1-x^2)w)$, $E_{n+1}(\cdot,(1-x^2)w)$ *and* $J_n(\cdot,(1-x^2)w)$, $E_{n+1}(\cdot,(1-x^2)w)$ *and* $p_n^{(1)}(\cdot,w)$, *and* $E_{n+1}(\cdot,(1-x^2)w)$ *and* $E_n(\cdot,(1-x^2)w)$, *respectively, separate each other.*

(b) *Suppose that* w *satisfies the assumptions* (4.5) *of Theorem* 4.1(b) *and let* $d_{n+1,n}$ *be defined by* (4.1). *Then there exists an* n_0 *such that for* $n\geqslant n_0$ *and* $\eta\in(0,(\xi_2-\xi_1)/2)$ *there is an odd number of zeros of* $J_n(\cdot,(1-x^2)w)$ *between two consecutive zeros of* $p_{n+1}(\cdot,w)$ *contained in* $[\xi_1+\eta,\xi_2-\eta]$, $\eta>0$, *and, if in addition* $\lim_{n\to\infty} d_{n+1,n}=0$, *there is an odd number of zeros of* $E_{n+1}(\cdot,(1-x^2)w)$ *between two consecutive zeros of* $p_n(\cdot,(1-x^2)w)$ *contained in* $[\xi_1+\eta,\xi_2-\eta]$.

Proof. We have by Theorem 4.1 combined with (4.7), (1.29), (1.31), and (2.7), $x=\cos\theta$,

$$\begin{aligned}
&\kappa 2^n E_{n+1}(x,(1-x^2)w)\,A_{n+1}\,p_{n+1}(x,w)\\
&\qquad +(1-x^2)\,2^n\kappa J_n(x,(1-x^2)w)\,B_{n+1}\,p_n(x,(1-x^2)w)\\
&\quad =\operatorname{Re}\{e^{-i(n+1)\theta}D(e^{i\theta})^{-1}+\varepsilon_{n,1}(e^{i\theta})\}\\
&\qquad \times\operatorname{Re}\{e^{-i(n+1)\theta}D(e^{i\theta})^{-1}+\varepsilon_{n,2}(e^{i\theta})\}\\
&\qquad +\operatorname{Im}\{e^{-i(n+1)\theta}D(e^{i\theta})^{-1}+\varepsilon_{n,1}(e^{i\theta})\}\\
&\qquad \times\operatorname{Im}\{e^{-i(n+1)\theta}D(e^{i\theta})^{-1}+\varepsilon_{n,2}(e^{i\theta})\}\\
&\quad =|D(e^{i\theta})|^{-2}+\eta_n(\theta),
\end{aligned}\tag{4.20}$$

where $\lim_{n\to\infty}\eta_n(\theta)=0$ uniformly on $[-\pi,\pi]$. Note that the last equation follows by simple calculation. Using (4.20) at the zeros of

$(x^2-1)\,p_n(\cdot,(1-x^2)w)$, respectively at the zeros of $p_{n+1}(\cdot, w)$, and recalling the known fact that the zeros of these both polynomials separate each other, the first and second statements follow. Using (4.20) at the zeros of $J_n(\cdot,(1-x^2)w)$ and taking into consideration the interlacing property of the zeros of $J_n(\cdot,(1-x^2)w)$ and $p_{n+1}(\cdot, w)$, the interlacing property of the zeros of $E_{n+1}(\cdot,(1-x^2)w)$ and $J_n(\cdot,(1-x^2)w)$ follows.

Finally let us turn to the interlacing property of the zeros of $E_{n+1}(\cdot,(1-x^2)w)$ and $p_n^{(1)}(\cdot, w)$ and of $E_{n+1}(\cdot,(1-x^2)w)$ and $E_n(\cdot,(1-x^2)w)$. As we have shown in the proof of Theorem 4.1 the assumptions on $\mu'(\theta):=w(\cos\theta)\,|\sin\theta|$ imply that $F(e^{i\theta}, d\mu)\in C^1[-\pi,\pi]$ and $F(e^{i\theta}, d\mu)\neq 0$ on $[-\pi,\pi]$. Hence the measure μ^* defined in (1.13), is absolutely continuous on $[-\pi,\pi]$ with the property that, see (1.15), $(\mu^*)'(\theta)=\text{const.}\,w(\cos\theta)\,|\sin\theta|/|F(e^{i\theta}, d\mu)|^2$ is continuously differentiable on $[-\pi,\pi]$ and thus satisfies a Lipschitz condition on $[-\pi,\pi]$. Hence, by (2.7),

$$\psi_n^*(e^{i\theta}, d\mu)=D(e^{i\theta},\mu^*)^{-1}+\varepsilon_{n,3}(e^{i\theta}),$$

where $\lim_{n\to\infty}\varepsilon_{n,3}(e^{i\theta})=0$ uniformly on $[-\pi,\pi]$. Setting

$$D(z, d\mu^*)={}^*D(z)$$

we get by (1.34)

$$B_{n+1}\sin\theta p_n^{(1)}(\cos\theta, w)=2\,\mathrm{Im}\{e^{i(n+1)\theta}({}^*D(e^{i\theta})+\varepsilon_{n,3}(e^{i\theta}))\}.$$

Now we claim that

$$\begin{aligned}\kappa 2^n E_{n+1}(x,(1-x^2)w)\,2^{-1}B_n\,p_{n-1}^{(1)}(x, w)&\\ -\kappa 2^{n-1}E_n(x,(1-x^2)w)\,2^{-1}\,B_{n+1}\,p_n^{(1)}(x, w)&\\ =(-1/c_0)+\gamma_n(x),&\end{aligned}\tag{4.21}$$

where $\lim_{n\to\infty}\gamma_n(x)=0$ uniformly on $[-1,1]$. Indeed, since the left hand side in (4.21) is equal to ($x=\cos\theta$)

$$\begin{aligned}&\mathrm{Re}\{e^{-i(n+1)\theta}(D(e^{i\theta})^{-1}+\varepsilon_{n,1}(e^{i\theta}))\}\\ &\quad\times\mathrm{Im}\{e^{in\theta}({}^*D(e^{-i\theta})+\varepsilon_{n,3}(e^{i\theta}))\}/\sin\theta\\ &\quad-\mathrm{Re}\{e^{-in\theta}(D(e^{i\theta})^{-1}+\varepsilon_{n,1}(e^{i\theta}))\}\\ &\quad\times\mathrm{Im}\{e^{i(n+1)\theta}({}^*D(e^{-i\theta})+\varepsilon_{n,3}(e^{i\theta}))\}/\sin\theta\\ &=-\mathrm{Re}\{(D+\varepsilon_{n,1})(e^{i\theta})\overline{({}^*D+\varepsilon_{n,3})(e^{i\theta})}\}\\ &=(-1/c_0)+\gamma_n(\cos\theta),\end{aligned}$$

where the first equality follows by simple calculation and the second one follows by the fact that in view of (1.11), by taking the limits there,

$$\operatorname{Re}\{D(e^{i\theta})\,\overline{{}^*D(e^{i\theta})}\} = 1/c_0.$$

Since it is well known that $p_n^{(1)}(\cdot, w)$ and $p_{n-1}^{(1)}(\cdot, w)$ have interlacing zeros we get from (4.21) that $E_{n+1}(\cdot, (1-x^2)w)$ and $p_n^{(1)}(\cdot, w)$ have interlacing zeros for $n \geqslant n_0$ and thus again by (4.21) $E_{n+1}(\cdot, (1-x^2)w)$ and $E_n(\cdot, (1-x^2)w)$ have interlacing zeros.

(b) follows immediately from relation (4.21) and Theorem 4.1(b). ∎

THEOREM 4.2. (a) *Suppose that w satisfies the assumptions* (4.2). *Then all quadrature weights $\sigma_{\nu,n}$, $\nu = 1, ..., n$, and $\gamma_{\mu,n}$, $\mu = 1, ..., n$, of the Gauss–Kronrod quadrature formula* (1.41), *where $\alpha'(x) = (1-x^2)\, w(x)$, are positive.*

(b) *Suppose that w satisfies the assumptions* (4.5) *and that $\lim_{n\to\infty} d_{n+1,n} = 0$. For any $\eta \in (0, (\xi_2 - \xi_1)/2)$ put $\mathcal{N} = \{\nu \in \{1, ..., n\} : x_{\nu,n} \in [\xi_1 + \eta, \xi_2 - \eta]\}$ and $\mathcal{M} = \{\mu \in \{1, ..., n+1\} : y_{\mu,n} \in [\xi_1 + \eta, \xi_2 - \eta]\}$, where $y_{\mu,n}$, respectively $x_{\nu,n}$, denotes the zeros of $E_{n+1}(\cdot, (1-x^2)w)$, respectively $p_n(\cdot, (1-x^2)w)$. Then the quadrature weights $\sigma_{\nu,n}$, $\nu \in \mathcal{N}$, and $\gamma_{\mu,n}$, $\mu \in \mathcal{M}$, of the Gauss–Kronrod quadrature formula* (1.41) *are positive.*

Proof. Since the proof runs along the same lines as the proof of Theorem 3 in our paper [24] we only sketch the proof.

(a) In view of [16] the positiveness of the $\gamma_{\mu,n}$'s is equivalent to the interlacing property of the zeros of $E_{n+1}(\cdot, (1-x^2)w)$ and $p_n(\cdot, (1-x^2)w)$. But this has been proved in Corollary 4.3.

Concerning the positiveness of the $\sigma_{\nu,n}$'s let us first recall that by [16]

$$\sigma_{\nu,n} = \frac{-1}{k_n^2 E_{n+1}(x_{\nu,n}; (1-x^2)w)\, P_n(x_{\nu,n}; (1-x^2)w)} \times \left(\frac{E_{n+1}(x_{\nu,n}; (1-x^2)w)}{P_{n+1}(x_{\nu,n}; (1-x^2)w)} - 1\right),$$

where k_n denotes, as in Corollary 4.1, the leading coefficient of $p_n(x, (1-x^2)w)$. Hence, taking into account the interlacing property of the zeros of $E_{n+1}(\cdot, (1-x^2)w)$ and $p_n(\cdot, (1-x^2)w)$, the positiveness of the $\sigma_{\nu,n}$'s is equivalent to

$$\frac{k_{n+1} E_{n+1}(x_{\nu,n}, (1-x^2)w)}{p_{n+1}(x_{\nu,n}, (1-x^2)w)} > 1 \qquad \text{for} \quad \nu = 1, ..., n. \tag{4.22}$$

Observing that by (1.31), $x_{\nu,n} = \cos\theta_{\nu,n}$,

$$z\phi_{2n+1}(z, d\mu) = \phi_{2n+1}^*(z, d\mu) \qquad \text{for} \quad z = e^{i\theta_{\nu,n}}, \tag{4.23}$$

and thus, using (1.2) and (1.2′),

$$\begin{aligned}\phi_{2n+2}(z, d\mu) &= \phi^*_{2n+2}(z, d\mu)\\ &= (1-a_{2n+1})\,\phi^*_{2n+1}(z, d\mu) \qquad \text{for} \quad z = e^{i\theta_{\nu,n}},\end{aligned} \tag{4.24}$$

we get with the help of (1.31), (1.2), (1.2′), and (4.24), respectively (1.29) and (4.23), that the following two relations hold for $x = x_{\nu,n}$ and $z = e^{i\theta_{\nu,n}}$:

$$p_{n+1}(x, (1-x^2)w) = \frac{2\kappa_{2n+4} z^{-(n+1)}\phi^*_{2n+2}(z)}{B_{n+2}\kappa_{2n+3}}, \tag{4.25}$$

and

$$p_n(x, (1-x^2)w) = \frac{2z^{-(n+1)}\phi^*_{2n+1}(z)}{A_{n+1}}. \tag{4.26}$$

Taking into account that by (1.22′) and (2.1) (ϕ^*_n) is uniformly bounded from above and below on $|z| = 1$ and thus $(|p_{n+1}(x_{\nu,n}, w)|)$, respectively $(|p_{n+1}(x_{\nu,n}, (1-x^2)w)|)$, is uniformly bounded from above, respectively below, and that by (4.17) $\lim_{n\to\infty} (k_{n+1}/k_n) = 2$, we obtain with the help of (4.11) that

$$\begin{aligned}\frac{k_{n+1}E_{n+1}(x_{\nu,n}, (1-x^2)w)}{p_{n+1}(x_{\nu,n}, (1-x^2)w)} &= \frac{2p_{n+1}(x_{\nu,n}, w)}{p_{n+1}(x_{\nu,n}, (1-x^2)w)} + \varepsilon_n(x_{\nu,n})\\ &= 2 + \tilde{\varepsilon}_n(x_{\nu,n})\end{aligned}$$

with $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and $\tilde{\varepsilon}_n \xrightarrow[n\to\infty]{} 0$ uniformly on $[-1, 1]$, where the last equality follows by (4.24)–(4.26), (2.5), (1.28), and (1.1). Thus part (a) is proved.

Using Corollary (4.3b) part (b) can be proved quite similarily to part (a). ∎

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